

QUATERNIONIC KÄHLER REDUCTIONS OF WOLF SPACES

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Abstract: The main purpose of the following article is to introduce a *Lie theoretical* approach to the problem of classifying pseudo quaternionic-Kähler (QK) reductions of the pseudo QK symmetric spaces, otherwise called *generalized Wolf spaces*.

The history of QK geometry starts with the celebrated *Berger's Theorem* [Ber55] which classifies all the irreducible holonomy groups for not locally symmetric pseudo-riemannian manifolds. In fact, a pseudo QK manifold (M, g) of dimension $4n$ ($n > 1$) is traditionally defined by the reduction of the holonomy group to a subgroup of $\mathrm{Sp}(k, l)\mathrm{Sp}(1)$ ($k + l = n$). Alekseevsky proved [Ale75] that any pseudo QK manifold is Einstein and satisfies some additional curvature condition, so it is possible to extend in a natural way the definition of QK manifolds to 4-manifolds: an oriented 4-manifold is said to be QK if it is Einstein and self-dual. Furthermore, the whole definition can be naturally extended to orbifolds [GL88].

Examples of pseudo QK manifolds are not too many, and most of them are homogeneous spaces. In particular, Alekseevsky proved that all homogeneous, Riemannian QK manifolds with positive scalar curvature are (compact and) symmetric [Ale75]. These spaces were classified by Wolf [Wol65] and have been called *Wolf spaces*.

The Wolf spaces together with their duals can be characterized as all the non-scalar flat QK manifolds admitting a transitive unimodular group of isometries [AC97].

The duals of Wolf spaces don't classify all the homogeneous, Riemannian QK manifolds with negative scalar curvature: in fact, there are more such spaces, like the so-called *Alekseevskian spaces* [Ale75].

Most examples of non-homogeneous QK *orbifolds* emerge via the so-called **symmetry reduction**. This process can be seen as a variation of a well-known construction of Marsden and Weinstein developed in the context of Poisson and symplectic geometry (see [MW74] or [AM78]). The Marsden-Weinstein quotient considers a symplectic manifold with some symmetries. A *new* symplectic manifold of lower dimension and fewer symmetries is then obtained by “dividing out” some symmetries in a “symplectic fashion”. This simple idea has been more recently applied in many different geometric situations. Various generalizations of the symplectic reductions include Kähler quotients, hyperkähler and hypercomplex quotients, quaternionic Kähler and quaternionic quotients, 3-Sasakian, Sasakian and contact quotients to mention just a few. The so-called **QK reduction** has been introduced by Galicki and Lawson (see [Gal86, GL88]). Here one starts with a QK space M^{4n} with some symmetry H . A new QK space of dimension $4n - 4\dim(H)$ is constructed out of the quaternionic Kähler moment map.

In this paper we specialize the QK-reduction to pseudo QK symmetric spaces, that we call *generalized Wolf spaces* (or sometimes just *Wolf spaces*). Recently, this spaces have been classified by Alekseevsky and Cortés [AC05].

In particular, we show that given a (generalized) Wolf space G/H the quaternionic Kähler moment map can be lifted to a \mathbb{R}^3 -valued map defined on the group G .

In the first two sections of the paper we briefly introduce the pseudo QK geometry and the fundamental notion of *diamond diagram* \diamond , a “bundle diagram” built up by Boyer, Galicki and Mann [BGM93] which functorially relates pseudo QK structures to pseudo 3-Sasakian, pseudo hyperkähler and pseudo twistor structures, studied respectively by Konishi [Kon75], Swann [Swa91] and Salamon [Sal82].

In the third section we sketch the QK reduction and show that the QK moment map is canonically associated to a moment map of the whole diamond diagram [BGM93, Swa91].

In the fourth section we introduce the Wolf spaces and describe their associate diamond diagrams.

In the fifth and sixth sections we give the explicit formula for the “diamond moment map” associated to Wolf spaces, though as a \mathbb{R}^3 -valued map defined on the total group G . We discuss its properties and focus our attention to 1-dimensional actions: in this case, we show that the classification of reductions is related to the problem to classify the adjoint orbits of the total group.

In the last part of the paper we apply the whole construction to the Grassmann manifold $\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)$, together its dual space $\mathrm{SO}_0(3,4)/\mathrm{SO}(3) \times \mathrm{SO}(4)$. Firstly, we find “canonical” reductions of them, related to some *normed algebra* structures of \mathbb{R}^8 , i.e. the reductions

$$\diamond(\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)) \xrightarrow{\mathrm{U}(1)} \diamond(\mathbb{Z}_3 \setminus G_2/\mathrm{SO}(4))$$

$$\diamond(\mathrm{SO}_0(3,4)/\mathrm{SO}(3) \times \mathrm{SO}(4)) \xrightarrow{\mathrm{U}(1)} \diamond(\mathbb{Z}_3 \setminus G_{2(2)}/\mathrm{SO}(4))$$

Next, we classify the adjoint orbits of $\mathrm{SO}(7)$ and $\mathrm{SO}(3,4)$.

Since the compactness of $\mathrm{SO}(7)$, the classification of its adjoint orbits is elementary and the reductions are nothing but the “weighted deformations” of the canonical reduction.

In the case of $\mathrm{SO}(3,4)$, the classification of the adjoint orbits is much more complicated and, besides the weighted deformations of the canonical reduction, we obtain many other reductions which are always *smooth*.

1. INTRODUCTION

Let \mathcal{O}^{4n} denote a $4n$ -dimensional (connected) manifold (or orbifold). An **almost quaternionic structure** on \mathcal{O}^{4n} is a rank 3 bundle $\mathfrak{H} \subset \mathrm{End}(T\mathcal{O})$ such that at every point there are local sections J_1, J_2, J_3 of \mathfrak{H} satisfying

$$J_a J_b = -\delta_{ab} + \epsilon^{abc} J_c.$$

It follows that \mathcal{O}^{4n} is almost quaternionic if and only if the structure group of the tangent bundle reduces to $\mathrm{GL}(n, \mathbb{H})\mathbb{H}^* = \mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$. Given a pseudoriemannian metric g on \mathcal{O}^{4n} , the pair (\mathfrak{H}, g) is called **pseudo-hyperhermitian structure** on \mathcal{O}^{4n} if the bundle \mathfrak{H} defines an almost quaternionic structure as defined above and \mathfrak{H} is contained in the vector subbundle of $\mathrm{End}(T\mathcal{O})$ generated by the isometries of g :

$$\mathfrak{H} \subset \mathbb{R} \cdot \mathrm{O}_g(T\mathcal{O}).$$

In terms of the local frame J_1, J_2, J_3 , this means that

$$g(J_a X, J_a Y) = g(X, Y)$$

for any local vector fields X, Y and for all $a = 1, 2, 3$. In particular, the signature of g has to be of the type $(4k, 4l)$, $k + l = n$.

Remark 1.1. *Given a pseudo-hyperhermitian structure (\mathfrak{H}, g) , one can use the local duality $J_a \xleftrightarrow{g} \omega_a$ to identify \mathfrak{H} with a subbundle of $\bigwedge^2 T^*M$. In particular, the local forms ω_a are almost symplectic.*

If $n > 1$, \mathcal{O}^{4n} is said to be **quaternionic-Kähler (QK)** with respect to the pseudo-hyperhermitian structure (\mathfrak{H}, g) if the bundle \mathfrak{H} is preserved by the Levi-Civita connection of g .

Theorem 1.2. *Let \mathcal{O}^{4n} as above ($n \geq 1$) and let g be a pseudoriemannian metric on \mathcal{O} with signature $(4k, 4l)$. Then, there exists an almost quaternionic structure \mathfrak{H} on \mathcal{O} such that (\mathfrak{H}, g) is a pseudo-hyperhermitian structure if and only if the tangent bundle reduces to the group $\mathrm{Sp}(k, l)\mathrm{Sp}(1)$. Moreover, if $n > 1$, \mathcal{O}^{4n} is QK with respect to (\mathfrak{H}, g) if and only if*

$$\mathrm{Hol}(g) \subset \mathrm{Sp}(k, l)\mathrm{Sp}(1)$$

Given a pseudo-hyperhermitian structure (\mathfrak{H}, g) on \mathcal{O}^{4n} , we can define two global tensors, locally written as

$$(1) \quad \Theta := \sum_a \omega_a \otimes J_a \in \Gamma(\bigwedge^2 T^* \mathcal{O} \otimes \mathfrak{H}),$$

$$(2) \quad \Omega := \sum_a \omega_a \wedge \omega_a \in \Gamma(\bigwedge^4 T^* \mathcal{O}).$$

where ω_a are local almost symplectic forms, dual to J_a via the metric g .

Theorem 1.3. *Let (\mathfrak{H}, g) be a pseudo-hyperhermitian structure on \mathcal{O}^{4n} . If $n > 1$, then the following conditions are equivalent:*

- (1) \mathcal{O}^{4n} is QK w.r.t. (\mathfrak{H}, g) .
- (2) Θ is parallel.
- (3) Ω is parallel.

It follows that the bundle \mathfrak{H} is orientable and the pseudoriemannian metric g induces a metric on \mathfrak{H} , given by

$$\langle J, J' \rangle := -\frac{1}{4n} \mathrm{trace}_g(J \circ J')$$

which is clearly positive defined. We can introduce the bundle of the oriented orthogonal frames of \mathfrak{H}

$$\pi : \mathcal{S} \rightarrow \mathcal{O},$$

called the **Konishi bundle** [Kon75] of the pseudo-hyperhermitian structure (\mathfrak{H}, g) . If $n > 1$ and \mathcal{O}^{4n} is QK relative to (\mathfrak{H}, g) , then the Levi-Civita connection induced on \mathfrak{H} is metric relative to \langle, \rangle , which means that

$$X \langle J, J' \rangle = \langle \nabla_X J, J' \rangle + \langle J, \nabla_X J' \rangle.$$

Moreover, the local coefficients arising from the formula

$$\nabla J_i = \alpha_i^j \otimes J_j$$

(or, equivalently, $\nabla \omega_i = \alpha_i^j \omega_j$ by duality) define a connection on the $\mathrm{SO}(3)$ -principal bundle \mathcal{S} , whose curvature F is given by

$$F_i^j := d\alpha_i^j - \alpha_i^k \wedge \alpha_k^j.$$

The curvature F is related to the Riemann curvature R , seen as a map

$$R : \bigwedge^2 T\mathcal{O} \rightarrow \mathrm{SkewEnd}(T\mathcal{O}).$$

We get

$$[R, J_i] = F_i^j \otimes J_j.$$

As the image of R takes values in $\mathfrak{sp}(k, l) \oplus \mathfrak{sp}(1)$, we must have

$$[R, J_i] = [R_{\mathfrak{sp}(1)}, J_i],$$

and, hence,

$$R_{\mathfrak{sp}(1)} = \frac{1}{2}(F_2^3 \otimes J_1 + F_3^1 \otimes J_2 + F_1^2 \otimes J_3).$$

Furthermore, it turns out that

$$F_i^j = \lambda \epsilon^{ijk} \omega_k$$

for some constant λ which gives

$$(3) \quad R_{\mathfrak{sp}(1)} = \frac{\lambda}{2} \Theta$$

Finally, the condition $[R_{\mathfrak{sp}(n)}, J_i] = 0$ implies

$$\mathrm{Ric}_g(X, Y) = \mathrm{trace}_g \{u \mapsto R_{\mathfrak{sp}(1)}(u, X)Y\} = -\frac{3\lambda}{2}g(X, Y).$$

In particular, the metric g is Einstein (see also [Ale68] or [Sal82]).

Let us now discuss the 4-dimensional case: clearly, \mathcal{O}^4 admits a pseudo-hyperhermitian structure if and only if it is oriented. In this case, if (g, \mathfrak{H}) is a pseudo-hyperhermitian structure on \mathcal{O}^4 , then $\pm g$ is a Riemannian metric and \mathfrak{H} is canonically identified with the bundle Λ_+ of the self-dual 2-forms. Moreover, the holonomy is contained in $\mathrm{SO}(4) = \mathrm{Sp}(1)\mathrm{Sp}(1)$, and the identities $\nabla \Omega = \nabla \Theta = 0$ are trivially satisfied.

Definition 1.4. *Let (g, Λ_+) be a pseudo-hyperhermitian structure on \mathcal{O}^4 . Then, \mathcal{O}^4 is QK with respect to (g, Λ_+) if g is Einstein and the condition (3) is satisfied.*

In particular, a QK-manifold has constant scalar curvature and the scalar-flat QK manifolds are *locally* hyperkähler.

From this point on \mathcal{O} will always be a non scalar-flat QK space of dimension greater or equal than 4 and signature $(4k, 4l)$.

2. THE DIAMOND DIAGRAM

2.1. The 3-Sasakian structure of the Konishi bundle. Let $\eta = (\eta_1, \eta_2, \eta_3)$ be the connection 1-form of the Konishi bundle $\pi : \mathcal{S} \rightarrow \mathcal{O}$, defined locally by the forms α_i^j . It can be shown that the forms η_i are indeed *contact forms* and they define uniquely three global vector fields ξ_1, ξ_2, ξ_3 such that

$$\eta_i(\xi_i) \equiv 1, \quad i_{\xi_i} d\eta_i = 0.$$

Furthermore, one sees that the ξ_i 's define a global 3-frame on \mathcal{S} (called **Reeb 3-distribution**), satisfying the rule

$$[\xi_a, \xi_b] = 2\epsilon_{abc}\xi_c$$

which implies the integrability. Note that the Reeb distribution is the vertical distribution of the Konishi bundle.

Let $g_{\mathcal{O}}$ be the metric on \mathcal{O} , and define the following pseudoriemannian metric (with signature $(4k+3, 4l)$) on \mathcal{S} :

$$g_{\mathcal{S}} := \pi^* g_{\mathcal{O}} + \sum_i \eta_i \otimes \eta_i.$$

Since ξ_i are unit Killing vector fields w.r.t. $g_{\mathcal{S}}$, the tensors

$$\Theta_i := \nabla \xi_i$$

are skewsymmetric, and they satisfy

- $\Phi_i^2 = -\text{Id} + \eta_i \otimes \xi_i$,
- $g_{\mathcal{S}} \circ (\Phi_i \otimes \Phi_i) = g_{\mathcal{S}} - \eta_i \otimes \eta_i$,
- $g_{\mathcal{S}} \circ (\Phi_i \otimes \text{Id}) = d\eta_i$.

In particular, (η_i, ξ_i, Φ_i) are *contact metric structures* w.r.t. $g_{\mathcal{S}}$. Consider the **metric cone** on \mathcal{S} (sometimes called the **Swann's bundle** [Swa91] of \mathcal{O} and denoted by $\mathcal{U}(\mathcal{O})$), i.e.

$$(\mathcal{C}(\mathcal{S}), g_{\mathcal{C}}) := (\mathcal{S} \times \mathbb{R}^+, t^2 g_{\mathcal{S}} + dt^2).$$

The cone $\mathcal{C}(\mathcal{S})$ is endowed with three almost complex structures

$$J_i : \begin{cases} J_i(X) &= \Phi_i(X) - \eta_i(X)\Psi \\ J_i(\Psi) &= \xi_i \end{cases}, \quad X \in \Gamma(T\mathcal{S}), \quad \Psi = t\partial_t.$$

Indeed, $(J_1, J_2, J_3, g_{\mathcal{C}})$ is a **pseudo-hyperkähler structure** (with signature $(4k+4, 4l)$) on \mathcal{C} . In other words, $(\{\eta_i, \xi_i, \Phi_i\}_{i=1}^3, g_{\mathcal{S}})$ is a **pseudo-3-Sasakian structure** on \mathcal{S} . In particular, the metric $g_{\mathcal{S}}$ is Einstein.

2.2. The twistor space. Pick any element $\tau \in S^2 \subset \mathfrak{so}(3)$. The circle

$$S_{\tau}^1 \subset \text{SO}(3)$$

generated by τ acts freely and isometrically on \mathcal{S} , so we can define the **twistor space** [Sal82] $(\mathcal{Z}_{\tau}, g_{\mathcal{Z}_{\tau}})$, where $\mathcal{Z}_{\tau} := S_{\tau}^1 \setminus \mathcal{S}$ and $g_{\mathcal{Z}_{\tau}}$ is the metric (with signature $(4k+2, 4l)$) obtained from $g_{\mathcal{S}}$ via the quotient.

If $\tau = (a_1, a_2, a_3)$, define

$$\begin{aligned} \eta_{\tau} &:= a_1\eta_1 + a_2\eta_2 + a_3\eta_3, \\ \Phi_{\tau} &:= a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3, \\ \mathcal{D}_{\tau} &:= \ker \eta_{\tau}. \end{aligned}$$

The pair $(\mathcal{D}_\tau, \Phi_\tau)$ defines an S_τ^1 -invariant *CR-structure* on \mathcal{S} , hence an almost complex structure I_τ on \mathcal{Z}_τ . In fact, it turns out that $(\mathcal{Z}_\tau, I_\tau, g_{\mathcal{Z}_\tau})$ is pseudo-Kähler.

Finally, if $\tau = \tau_1 \times \tau_2$ we can define the following \mathbb{C} -valued 1-form on \mathcal{S} , namely

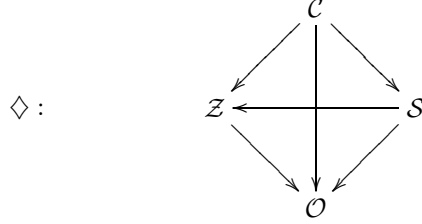
$$\Upsilon_\tau^{\tau_1, \tau_2} := \eta_{\tau_1} + i\eta_{\tau_2}.$$

Furthermore, the complex line bundle L_τ generated by $\Upsilon_\tau^{\tau_1, \tau_2}$ doesn't depend on τ_1, τ_2 and it is S_τ^1 -invariant, so it pushes down to a bundle

$$\mathcal{L}_\tau \subset T^{(1,0)}\mathcal{Z}_\tau$$

which defines a **pseudo-holomorphic contact structure** on \mathcal{Z}_τ , whose isomorphism class doesn't depend on the choice of $\tau \in S^2$. In particular, the biholomorphism class of \mathcal{Z}_τ doesn't depend on $\tau \in S^2$.

Putting all these structures together, we obtain the following diagram:



3. THE REDUCED DIAMOND DIAGRAM

Let \mathcal{O} be a QK manifold with a group of isometries T preserving the bundle \mathfrak{H} . Relative to this action, we can associate the notion of a **moment map** (see [Gal87] and [GL88]), which is a *section*

$$\mu : \mathcal{O} \rightarrow \mathfrak{t}^* \otimes \mathfrak{H},$$

$$x \mapsto \{v \mapsto \mu_v(x)\},$$

where \mathfrak{t} is the Lie algebra of T . The moment map has the following properties:

- it is T -invariant, i.e.

$$\mu_{\text{Ad}_t v}(tx) = \mu_v(x)$$

- it satisfies the equation

$$\nabla \mu_v = i_{v_{\mathcal{O}}} \Theta,$$

where $v_{\mathcal{O}}$ is the vector field induced on \mathcal{O} by $v \in \mathfrak{t}$.

The moment map *always exists and it is unique* as it is completely determined by the second of the above conditions. In fact, we have ([Gal87]):

$$\mu_v = \frac{\lambda'}{s} \sigma(\mathcal{L}_{v_{\mathcal{O}}} - \nabla_{v_{\mathcal{O}}}),$$

where σ is the bundle isomorphism

$$\sigma : \text{SkewEnd}(\mathfrak{H}) \rightarrow \mathfrak{H}$$

such that

$$\sigma^{-1}(J) = \frac{1}{2} \text{ad}_J = J \wedge .$$

It can be proven that, under appropriate assumptions on the action and on the regularity of μ , the quotient $T \backslash\backslash \mathcal{O} := T \setminus \mu^{-1}(0)$ is a QK orbifold (see [GL88]). The whole process

$$\mathcal{O} \xrightarrow{T} T \backslash\backslash \mathcal{O}$$

is then called *QK-reduction* of \mathcal{O} .

3.1. Diamond reductions. Let $\phi : \mathcal{O} \rightarrow \mathcal{O}$ be a diffeomorphism preserving the QK structure $(\mathfrak{H}, g_{\mathcal{O}})$, i.e.,

$$\phi^* g = g, \quad \phi^* \mathfrak{H} = \mathfrak{H}.$$

Given a local frame $\sigma = \{J_1, J_2, J_3\} \in \Gamma(\mathcal{S})$, we have

$$\phi_* J_i := (\phi^*)^{-1} J_i := \phi_* \circ J_i \circ \phi_*^{-1} = \beta_i^j J_j,$$

where $\beta_\sigma := \{\beta_i^j\}$ is a locally defined map with values in $\text{SO}(3)$, so ϕ induces a bundle automorphism which we denote by $\phi_{\mathcal{S}}$. There exists a map

$$\beta_\phi : \mathcal{S} \rightarrow \text{SO}(3)$$

such that $\beta \circ \sigma = \beta_\sigma$ for any local frame σ or, in other words,

$$\phi_{\mathcal{S}}(f) = (\beta_\phi(f)) \cdot f.$$

Furthermore, if $\alpha_\sigma = \{\alpha_i^j\}$ is defined by $\nabla J_i = \alpha_i^j \otimes J_j$, it follows that

$$\alpha_\sigma = \phi^*(\alpha_{(\beta_\phi(\sigma)) \cdot \sigma}) = \phi^*(\alpha_{\phi_{\mathcal{S}} \circ \sigma}).$$

Hence,

$$\phi_{\mathcal{S}}^* \eta = \eta$$

and

$$\phi_{\mathcal{S}}^* g_{\mathcal{S}} = g_{\mathcal{S}}.$$

Since $\phi_{\mathcal{S}}$ preserves the 3-Sasakian structure, it is easy to see that it induces an automorphism of the twistor space. Finally, the automorphism of the cone is obtained by trivially lifting $\phi_{\mathcal{S}}$.

Let $\pi : P \rightarrow \mathcal{O}$ be any one of the three bundles in the diamond diagram. If μ is the QK-moment map on \mathcal{O} and X is a Killing vector field on P , define a map

$$\tilde{\mu}_X := \mu_{\pi_* X} \circ \pi$$

which is a section of the pull-back bundle $\pi^* \mathfrak{H}$.

From the axioms of the QK moment map, it follows that

$$\nabla \tilde{\mu}_X = i_X \pi^* \Theta$$

Lemma 3.1. *For $P = \mathcal{S}$ or \mathcal{C} , the bundle $\pi^* \mathfrak{H}$ is trivial. Moreover,*

- *if $P = \mathcal{S}$, then $\tilde{\mu} = \eta$;*
- *if $P = \mathcal{C}$, then $\tilde{\mu}$ is an hyperkähler moment map.*

The map η is called **3-Sasakian moment map**, and the map induced on \mathcal{Z} (for any fixed Killing field) is a section of the bundle $\mathbb{R} \oplus \mathcal{L}$ and is called **twistor moment map**.

We conclude the following: If T is a group of QK isometries acting on \mathcal{O} then it induces an action on the whole diamond diagram preserving all the relevant structures, and the QK moment map can be extended to a moment map of the whole \diamond . Consequently it makes more sense to talk about the **diamond reduction**:

$$\diamond \xrightarrow{\dots\dots\dots T} T \\\diamond .$$

It can be proven that, under appropriate assumptions, the reduced diamond diagram $T \\\diamond$ is the diamond diagram of the QK orbifold $T \\\mathcal{O}$ [Swa91, BGM94].

4. WOLF SPACES

The classification of (pseudo-)riemannian symmetric spaces reduces to the study of involutive Lie algebras, i.e., Lie algebras endowed with a canonical splitting (of vector subspaces)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where \mathfrak{h} is a subalgebra and

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

In particular, any involutive Lie algebra defines corresponding homogeneous space $M = G/H$, where G, H are connected Lie groups (generated by $\mathfrak{g}, \mathfrak{h}$ respectively), and G is simply connected.

The canonical splitting defines an ℓ_G -invariant connection on the principal bundle $\pi : G \rightarrow G/H$, given by

$$T_g G = l_{g*} \mathfrak{h} \oplus l_{g*} \mathfrak{m}$$

Via the horizontal lifting, for any open set $V \subset M$ we construct a linear isomorphisms between sections and equivariant maps

$$\Gamma_V(TM) \rightarrow C_H^\infty(\pi^{-1}(V), \mathfrak{m})$$

$$X \rightarrow X^*.$$

In particular, $TM = G \times_H \mathfrak{m}$. We have similar isomorphisms for other types of tensors: for example, the set of all the metrics which give M the structure of a pseudoriemannian symmetric space is the set Σ of all the Ad_H -pseudoscalar products on \mathfrak{m} .

Theorem 4.1. *Let ∇ be the linear connection on TM associated to the canonical splitting of \mathfrak{g} . Then ∇ is ℓ_G -invariant and is the Levi-Civita connection of TM with respect to all the elements of Σ . Furthermore, its curvature two-form R is associated to a map*

$$R^* : G \rightarrow \bigwedge^2 \mathfrak{m}^*$$

given by

$$R^*(v, w) = -[v, w]$$

Thus, by the Ambrose-Singer theorem, the holonomy algebra is given by $\text{ad}_{[\mathfrak{m}, \mathfrak{m}]}|_{\mathfrak{m}}$. The irreducibility of this algebra implies that \mathfrak{g} is semisimple. As a scalar product we can thus take the restriction of the Killing form of \mathfrak{g} to \mathfrak{m} . Motivated by all these facts we give the following

Definition 4.2. *A [generalized] Wolf space is a semisimple, involutive Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that*

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{s},$$

where the last splitting is the direct sum of two commuting Lie subalgebras, such that $\mathfrak{s} \simeq \mathfrak{sp}(1)$ and \mathfrak{h}' is isomorphic to a Lie subalgebra of $\mathfrak{sp}(l, k)$ with $4k + 4l = \dim \mathfrak{m}$.

It follows that a Wolf space is (pseudo-)quaternion-Kähler, and that the adjoint representation of $H = H'S$ (where $\mathfrak{s} = \text{Lie}(S)$, $\mathfrak{h}' = \text{Lie}(H')$) on \mathfrak{m} and the usual one are isomorphic. Moreover, the action of G on M is almost effective and the Killing algebra of M is \mathfrak{g} . The bundle

$$\mathfrak{H} := G \times_H \mathfrak{s}$$

defines the almost quaternionic hermitian structure, and we have the linear isomorphisms

$$\begin{aligned} \Gamma_V(\mathfrak{H}) &\rightarrow C_H^\infty(\pi^{-1}(V), \mathfrak{s}) \\ J &\mapsto J^* \end{aligned}$$

such that

$$(JX)^* = [J^*, X^*],$$

as long as X is left invariant. Moreover, the linear connection induced on \mathfrak{H} is the restriction of the Levi-Civita connection on tensors to \mathfrak{H} .

The 3-Sasakian bundle is given by

$$\mathcal{S} = G \times_H \text{SO}(3) = G/H' \times_{\text{SO}(3)} \text{SO}(3) = G/H'$$

with the trivial projection. The 1-form defining the 3-sasakian structure is the connection 1-form associated to the Levi-Civita connection of \mathfrak{H} , i.e.,

$$\eta(l_{g*}v) = v_{\mathfrak{s}}$$

for each $v \in \mathfrak{g}$.

Now, pick $i \in S^2 \subset \mathfrak{s}$. This induces a vector field and an U -action on \mathcal{S} , where $U \simeq \text{U}(1)$ is the centralizer of i in S and the action is nothing but the right multiplication. Therefore,

$$\mathcal{Z} = G/H'U$$

In conclusion, the diamond diagram is given by

$$\begin{array}{ccccc} & & \mathcal{C}(G/H') & & \\ & \swarrow & \downarrow & \searrow & \\ G/H'U & \leftarrow & & \rightarrow & G/H' \\ & \swarrow & \downarrow & \searrow & \\ & & G/H & & \end{array}$$

4.1. The Alekseevsky-Cortés's list. Recently, the generalized Wolf spaces have been classified (see [AC05]):

$$\begin{array}{cccc} \frac{\text{SU}(p+2, q)}{\text{S}(\text{U}(2) \times \text{U}(p, q))} & \frac{\text{SL}(n+1, \mathbb{H})}{\text{S}(\text{GL}(1, \mathbb{H}) \times \text{GL}(n, \mathbb{H}))} & & \\ \frac{\text{SO}_o(p+4, q)}{\text{SO}(4) \times \text{SO}_o(p, q)} & \frac{\text{SO}^*(2l+4)}{\text{SO}^*(4) \times \text{SO}^*(2l)} & & \\ \frac{\text{Sp}(p+1, q)}{\text{Sp}(1) \times \text{Sp}(p, q)} & & & \\ \frac{E_{6(-78)}}{\text{SU}(2)\text{SU}(6)} & \frac{E_{6(2)}}{\text{SU}(2)\text{SU}(6)} & \frac{E_{6(2)}}{\text{SU}(2)\text{SU}(2, 4)} & \frac{E_{6(-14)}}{\text{SU}(2)\text{SU}(2, 4)} \end{array}$$

$$\begin{array}{ccccc}
\frac{E_{6(6)}}{\mathrm{Sp}(1)\mathrm{SL}(3, \mathbb{H})} & \frac{E_{6(-26)}}{\mathrm{Sp}(1)\mathrm{SL}(3, \mathbb{H})} & & & \\
\frac{E_{7(-133)}}{\mathrm{SU}(2)\mathrm{Spin}(12)} & \frac{E_{7(-5)}}{\mathrm{SU}(2)\mathrm{Spin}(12)} & \frac{E_{7(-5)}}{\mathrm{SU}(2)\mathrm{Spin}_o(4, 8)} & \frac{E_{7(7)}}{\mathrm{SU}(2)\mathrm{SO}^*(12)} & \frac{E_{7(-25)}}{\mathrm{SU}(2)\mathrm{SO}^*(12)} \\
\frac{E_{8(-248)}}{\mathrm{SU}(2)E_{7(133)}} & \frac{E_{8(-24)}}{\mathrm{SU}(2)E_{7(133)}} & \frac{E_{8(-24)}}{\mathrm{SU}(2)E_{7(-5)}} & \frac{E_{8(8)}}{\mathrm{SU}(2)E_{7(-5)}} & \\
\frac{F_{4(-52)}}{\mathrm{Sp}(1)\mathrm{Sp}(3)} & \frac{F_{4(4)}}{\mathrm{Sp}(1)\mathrm{Sp}(3)} & \frac{F_{4(4)}}{\mathrm{Sp}(1)\mathrm{Sp}(1, 2)} & \frac{F_{4(-20)}}{\mathrm{Sp}(1)\mathrm{Sp}(1, 2)} & \\
\frac{G_{2(-14)}}{\mathrm{SO}(4)} & \frac{G_{2(2)}}{\mathrm{SO}(4)} & & &
\end{array}$$

In the compact Riemannian case, we get the classical *classic Wolf spaces*, which arise from the classification of all QK homogeneous spaces with positive scalar curvature (see [Wol65]), namely the spaces

$$\begin{array}{ccccc}
\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)} & \frac{\mathrm{SU}(n)}{\mathrm{S}(\mathrm{U}(n-2) \times \mathrm{U}(1))\mathrm{Sp}(1)} & \frac{\mathrm{SO}(n)}{\mathrm{SO}(n-4) \times \mathrm{SO}(4)} & & \\
\frac{G_2}{\mathrm{SO}(4)} & \frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)} & \frac{E_6}{\mathrm{SU}(6)\mathrm{Sp}(1)} & \frac{E_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)} & \frac{E_8}{E_7\mathrm{Sp}(1)}
\end{array}$$

It is straightforward to verify that the set of generalized Wolf spaces is closed with respect to the duality of involutive Lie algebras, i.e. the map

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \longmapsto \mathfrak{h} \oplus i\mathfrak{m} \subset \mathfrak{g} \otimes \mathbb{C}$$

5. QUATERNIONIC REDUCTIONS OF WOLF SPACES

5.1. Formula for the moment map for Wolf spaces. Suppose we have now a Wolf space of the form $G/H'S$, and that the action is the left multiplication by T , where T is some *virtual* Lie subgroup of G (i.e., a subgroup of G , not necessarily closed). Then, it can be easily seen that this action preserves all the structures in the diamond diagram.

The aim of this work is to study (and eventually give some classification) of all the possible reductions we could get in this way, when T varies among all the virtual Lie subgroups of G .

Actually, in order to produce Hausdorff spaces, we'll require a *proper action* and in particular that T is indeed a Lie subgroup. For the moment we don't care and focus our attention on the Lie algebra which *generates the action*, i.e.

$$\mathfrak{t} := \mathrm{Lie}(T) \subset \mathfrak{g}$$

The advantage of working with (generalized) Wolf spaces is quite clear: The moment map has a particularly simple form. Up to some scalar factor, it is given by the formula

$$\mu_v(g) = (\mathrm{Ad}_g^{-1}v)_s$$

(seen as an equivariant map), for any $g \in G$ and $v \in \mathfrak{t}$.

(Note that the quaternionic \mathfrak{H} -valued two-form is given by

$$\Theta(v, w) = -\lambda[v, w]_s$$

for some $\lambda > 0$.)

As a matter of fact, this map *at the same time* represents the 3-sasakian and the quaternion-Kähler moment map, so we'll call it just *moment map*.

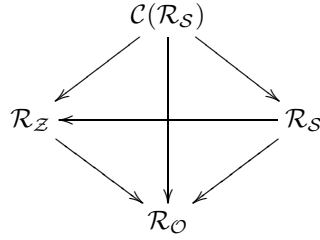
5.2. The reduced diamond diagram. We can define four *zero loci*:

$$\begin{aligned} Z_G(T) &:= \{g \in G : \mu_v(g) = 0 \text{ for each } v \in \mathfrak{t}\}, \\ Z_S(T) &:= \{gH' \in \mathcal{S} : \mu_v(g) = 0 \text{ for each } v \in \mathfrak{t}\} = Z_G/H', \\ Z_Z(T) &:= \{gH'U \in \mathcal{Z} : \mu_v(g) = 0 \text{ for each } v \in \mathfrak{t}\} = Z_G/H'U, \\ Z_O(T) &:= \{gH \in \mathcal{O} : \mu_v(g) = 0 \text{ for each } v \in \mathfrak{t}\} = Z_G/H, \end{aligned}$$

and the reductions

$$\mathcal{R}_G(T) := T \setminus Z_G(T), \quad \mathcal{R}_S(T) := T \setminus Z_S(T), \quad \mathcal{R}_Z(T) := T \setminus Z_Z(T), \quad \mathcal{R}_O(T) := T \setminus Z_O(T),$$

giving a diagram



Our formula of the moment map implies an *extra symmetry* with respect to the group G action, namely

$$Z_G(T^g) = g \cdot Z_G(T).$$

Hence, we can restrict our classification only to *conjugacy classes* of virtual Lie subgroups of G . Furthermore, it follows that the group $C_G(T)/T$ preserves the reduced diamond diagram.

Question 5.1. *Is this group the stabilizer of the diagram?*

There are a lot of problems in the reduction process. The reduced spaces may not have any orbifold structure or even fail to be Hausdorff. This, for example, can be due to some irregularity of the moment map or non-finiteness of some isotropy subgroups of T , the noncompactness of T , etc. Nevertheless, we shall examine examples (perhaps exceptional), where the moment map is not regular, some isotropies are not finite and one still can give to these spaces an orbifold structure in such a way that the above diagram is actually a diamond diagram.

6. 1-DIMENSIONAL TORIC REDUCTIONS

6.1. Singular and irregular points. Let us analyze the action of T on the zero loci. Note that it is automatically free on Z_G but on the other spaces we could get nontrivial isotropies. For instance, $tgK = gK$ if and only if $t \in T \cap K^g$ and in our cases we get three different kinds of isotropies (respectively, *3-Sasakian*, *twistor* and *quaternionic*), namely

$$T \cap (H')^g \subset T \cap (H'U)^g \subset T \cap (H)^g.$$

For each type of action we distinguish two different kinds of points with nontrivial isotropy, namely the *singular points* (with discrete isotropy) and *irregular points*

(all the other), so we have 3-Sasakian irregular and singular points...and so on.

Lemma 6.1. *If T is 1-dimensional, then Z_G contains only irregular points of 3-Sasakian type.*

Proof. Since T is 1-dimensional, it doesn't have nondiscrete, proper subgroups. Hence, the three irregularity conditions can be written in the following way:

$$\mathrm{Ad}_g^{-1}\mathfrak{t} \subset \mathfrak{h}',$$

$$\mathrm{Ad}_g^{-1}\mathfrak{t} \subset \mathfrak{h}' \oplus \mathfrak{u},$$

$$\mathrm{Ad}_g^{-1}\mathfrak{t} \subset \mathfrak{h}.$$

Comparing these conditions with the moment map equation

$$\mathrm{Ad}_g^{-1}\mathfrak{t} \subset \mathfrak{h}' \oplus \mathfrak{m}$$

proves the lemma. \square

For this reason, by *irregular points* we shall mean just irregular points of 3-Sasakian type.

6.1.1. *The regularity condition.* Now, let's write down the differential of the moment map: it can be thought as a map

$$d\mu_v : G \times \mathfrak{g} \rightarrow \mathfrak{s},$$

$$(g, w) \rightarrow d_g\mu_v(w).$$

An easy calculation shows that

$$d_g\mu_v(w) = -\mu_{[\mathrm{Ad}_g w, v]}(g) = -(\mathrm{Ad}_g^{-1}[\mathrm{Ad}_g w, v])_{\mathfrak{s}} = [\mathrm{Ad}_g^{-1}v, w]_{\mathfrak{s}}$$

Lemma 6.2. *If T is 1-dimensional then on Z_G the critical set of the moment map coincides with its irregular set and with the set of (3-Sasakian) irregular points.*

Proof: Let \langle, \rangle be the Killing form on \mathfrak{g} , $w \in \mathrm{Ker} d_g\mu_v$ for some $g \in Z_G$. Then, for each $\xi \in \mathfrak{g}$ we have

$$0 = \langle d_g\mu_v(w), \xi \rangle = \langle [\mathrm{Ad}_g^{-1}v, w]_{\mathfrak{s}}, \xi \rangle = -\langle w, [\mathrm{Ad}_g^{-1}v, \xi_{\mathfrak{s}}] \rangle.$$

Hence, $\mathrm{Ker} d_g\mu_v$ is the orthogonal complement of $[\mathrm{Ad}_g^{-1}v, \mathfrak{s}]$ which has codimension < 3 if and only if $(\mathrm{Ad}_g^{-1}v)_{\mathfrak{m}} = 0$. \square

In particular, this lemma says that, if the 3-Sasakian action is locally free, the normal bundle of Z_G in G is

$$\nu(Z_G) := \{(g, \xi) \in S \times \mathfrak{g} : g \in Z_G, \xi \in [\mathrm{Ad}_g^{-1}v, \mathfrak{s}]\} \simeq Z_G \times \mathfrak{s}.$$

Thus, Z_G is parallelizable. Moreover, the orthogonal projection

$$\pi : TG|_{Z_G} \rightarrow \nu(Z_G)$$

can be thought as a map

$$\pi : \mathfrak{g} \rightarrow \mathfrak{s}$$

satisfying

$$\pi(w) = \frac{\lambda}{\|(\mathrm{Ad}_g^{-1}v)_{\mathfrak{m}}\|} d_g\mu_v(w)$$

provided that the denominator doesn't vanish.

Again, π can be pushed down to a bundle map

$$\pi' : T\mathcal{O}|_{Z_{\mathcal{O}}} \rightarrow \nu(Z_{\mathcal{O}}).$$

Lemma 6.3. *The second fundamental form associated to the embedding $Z_{\mathcal{O}} \subset \mathcal{O}$ is given by the equivariant map*

$$\alpha : \alpha_g(w_1, w_2) = \frac{\lambda}{\|(\text{Ad}_g^{-1}v)_{\mathfrak{m}}\|} [[w_1, \text{Ad}_g^{-1}v], w_2]_{\mathfrak{s}}.$$

Proof: Let $X, Y : Z_G \rightarrow \mathfrak{m}$ be vector fields of $Z_{\mathcal{O}}$. The covariant derivative $\nabla_X Y$, thought as equivariant map, is given by

$$(\nabla_X Y)_g = (d_g Y)(l_{g*} X_g).$$

Thus, the second fundamental form is given by

$$\pi((d_g Y)(l_{g*} X_g)) = \frac{\lambda}{\|(\text{Ad}_g^{-1}v)_{\mathfrak{m}}\|} [\text{Ad}_g^{-1}v, (d_g Y)(l_{g*} X_g)]_{\mathfrak{s}}.$$

As

$$[\text{Ad}_g^{-1}v, Y_g]_{\mathfrak{s}} = 0 \quad \forall g \in Z_G$$

by taking $l_{g*} X_g \in T_g Z_G$, we get

$$0 = -[[X_g, \text{Ad}_g^{-1}v], Y_g]_{\mathfrak{s}} + [\text{Ad}_g^{-1}v, (d_g Y)(l_{g*} X_g)]_{\mathfrak{s}}.$$

□

We now are able to write the sectional curvature of $Z_{\mathcal{O}}$: it is given by

$$\mathfrak{K}_g(w_1, w_2) = \| [w_1, w_2] \| + \langle \alpha_g(w_1, w_1), \alpha_g(w_2, w_2) \rangle - \|\alpha_g(w_1, w_2)\|,$$

where the vectors w_1, w_2 are orthogonal.

6.2. The quaternion-Kähler structure of $\mathcal{R}_{\mathcal{O}}$. Suppose that the group T acts on Z_G with finite isotropy. Then, the bundle

$$T \setminus \nu(Z_G)/H = \mathcal{R}_G \times_H \mathfrak{s}$$

defines the quaternionic structure on $\mathcal{R}_{\mathcal{O}}$: in fact, it has fiber $\simeq \mathfrak{s}$ and can be thought as a subbundle of $\text{End}(T\mathcal{R}_{\mathcal{O}})$. Furthermore, the pull-back of the metric of \mathcal{O} on $Z_{\mathcal{O}}$ can be pushed to a metric on $\mathcal{R}_{\mathcal{O}}$.

6.3. The energy function. Related to the moment map we have the map

$$\begin{aligned} E : G &\rightarrow [0, +\infty), \\ g &\mapsto -\|\mu_v(g)\|, \end{aligned}$$

whose gradient is given by

$$(\text{grad } E)_g = -[(\text{Ad}_g^{-1}v)_{\mathfrak{s}}, (\text{Ad}_g^{-1}v)_{\mathfrak{m}}].$$

Its critical set is given by the union of Z_G and all the quaternionic irregular points in G . It turns out that this vector field is *complete* (see [Bat04]), so it defines a global flow on G . Furthermore, if the adjoint orbit of v doesn't intersect \mathfrak{h} (note that, if H is compact, at most one adjoint orbit satisfies this condition, while it doesn't exist if G is compact), Z_G is the critical set of E .

Unfortunately, we do not know if the gradient flow converges to Z_G when G is not compact:

Question 6.4. *Is the zero locus Z_G (resp. Z_S , $Z_{\mathcal{U}}$, $Z_{\mathcal{O}}$) a deformation retract of G (resp. S , \mathcal{U} , \mathcal{O})?*

7. $\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)$ AND $\mathrm{SO}_o(3,4)/\mathrm{SO}(3) \times \mathrm{SO}(4)$

Let us consider the spaces $\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)$ and $\mathrm{SO}_o(3,4)/\mathrm{SO}(3) \times \mathrm{SO}(4)$. The first one is the usual Grassmann manifolds of 4-planes in \mathbb{R}^7 , or in other words the space of all the orthogonal splittings

$$W^4 \oplus (W^4)^\perp = \mathbb{R}^7$$

with respect to the usual euclidean scalar product.

The latter space is (the connected component of some fixed point o of) the space of splittings as before, but with respect the standard *pseudo-euclidean scalar product with signature (3, 4)*:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 - dx_5^2 - dx_6^2 - dx_7^2,$$

and such that W^4 and $(W^4)^\perp$ are *space-like* and *time-like* subspaces, respectively (i.e., such that the restriction of the scalar product is respectively negative and positive defined).

We shall study all the possible homogeneous 1-dimensional quaternionic Kähler reductions. In order to do this, it will be useful to give a list of all the adjoint orbits of the group G . Actually, since the first group is compact the classification is elementary: all the adjoint orbits in $\mathrm{SO}(7)$ have a representative of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & -c & 0 \end{pmatrix}$$

where a, b, c are nonnegative real numbers.

The classification of adjoint orbits of $\mathrm{SO}(3,4)$ is much more complicated and gives several families of adjoint orbits, each of them depending on some set of parameters. In order to do that, we'll quote an algorithm found by Burgoyne and Cushman (see [BC77]), valid for all classical groups.

7.1. Octonions and split octonions. Before to start the classification of adjoint orbits, I want to consider for both of this spaces a particular reduction, which involves some considerations about *normed algebra* structures on \mathbb{R}^8 .

7.1.1. The Cayley-Dickson process. (See [Har90] for more details.) We first recall what octonions and split octonions are, starting from some general facts about normed algebras.

A normed algebra is a (real) algebra with unity V endowed with some non-degenerate quadratic form $\|\cdot\|$ such that $\|vw\| = \|v\|\|w\|$. Via polarization, this quadratic form can be identified with some pseudo-scalar product \langle, \rangle such that

$$\langle xz, yw \rangle + \langle xw, yz \rangle = 2 \langle x, y \rangle \langle z, w \rangle.$$

It follows that $\|1_V\| = 1$, so $\text{Re}(V) := \mathbb{R}1_V$ has an orthogonal complement in V which we denote with $\text{Im}(V)$. In a normed algebra we can define a *conjugation* by using this splitting in a natural way.

Now, given a normed algebra $(V, \|\cdot\|)$ we can construct two new algebras $V(\pm)$, whose underlying vector space is nothing but $V \oplus V$ and which contain V as a subalgebra. This construction (called *Cayley-Dickson process*) is motivated by the following

Lemma 7.1. *If A is a normed subalgebra (with $1 \in A$) of a normed algebra B and $\epsilon \in A^\perp$ is such that $\|\epsilon\| = \pm 1$, then ϵA is orthogonal to A and*

$$(a + \epsilon b)(c + \epsilon d) = (ac \mp \bar{d}b) + \epsilon(da + b\bar{c})$$

Hence, the product on $V(\pm)$ must be

$$(a, b)(c, d) = (ac \mp \bar{d}b, da + b\bar{c})$$

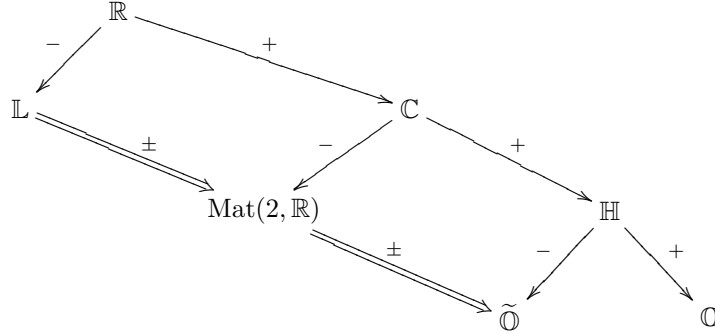
while the new norm is given by

$$\|(a, b)\| := \|a\| \pm \|b\|$$

The Cayley-Dickson process has the following properties:

- $V(\pm)$ is commutative if and only if $V = \mathbb{R}$;
- $V(\pm)$ is associative if and only if V is commutative and associative;
- $V(\pm)$ is normed iff $V(\pm)$ is alternative (i.e., the associator $[x, y, z]$ is alternating), iff V is associative.

In particular, if we start from \mathbb{R} we get the following diagram:



After the third step, this process doesn't produce other normed algebras. The well-known theorem due by Hurwitz says that these are the only possible normed algebras.

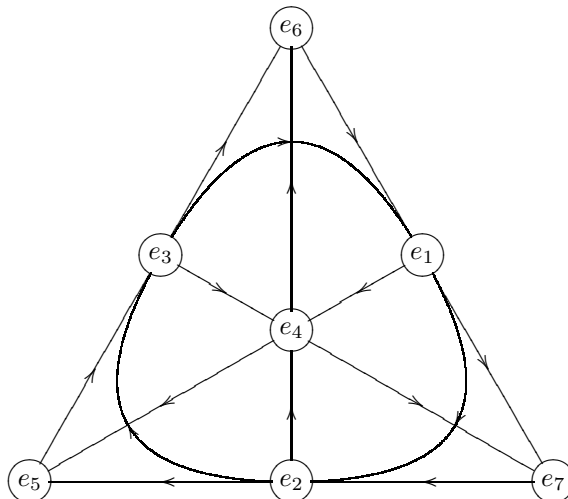
In particular, we want to focus our attention on the last two algebras \mathbb{O} (the *octonions*), $\tilde{\mathbb{O}}$ (the *split octonions*) and their groups of automorphisms.

Let's consider first the quaternions $\mathbb{H} = \langle 1, I_1, I_2, I_3 \rangle \simeq \mathbb{R}^4$ and define

$$1 = e_0 = (1, 0), \quad e_4 = (0, 1)$$

$$e_k = (I_k, 0), \quad e_{k+4} = (0, I_k)$$

The algebra structure of the octonions is completely described by a diagram (called **the Fano plane**), that with respect to the basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ has the form



In fact, if $e_i \neq e_j$ belong to the same edge of e_k , then $e_i e_j = \pm e_k$ where the sign depends on the order of e_i, e_j, e_k along the edge. Moreover, $e_i^2 = -1$ for each i .

Furthermore, with respect to this basis we obtain the multiplication rule of the split octonions by changing the sign to the products of all elements in $\{a_4, a_5, a_6, a_7\}$. In the case of octonions, the norm arising from the Cayley-Dickson process is the euclidean one, while in the case of split octonions it has signature $(4, 4)$.

Lemma 7.2. *Let $x, y, z \in \mathbb{O}$ (or $\in \tilde{\mathbb{O}}$) be purely imaginary and orthogonal. Then*

- $xy = -yx$ is purely imaginary and orthogonal to x, y ;
- the double product $x(yz)$ is alternating.

Lemma 7.3. *Let $x, y, z \in \mathbb{O}$ (or $\in \tilde{\mathbb{O}}$). Then we have the following Moufang identities:*

- $(xyx)z = x(y(xz))$;
- $z(xyx) = ((zx)y)x$;
- $(xy)(zx) = x(yz)x$.

Let $G_2, G_{2(2)} \subset \mathrm{GL}(8, \mathbb{R})$ be the automorphism groups of the algebras $\mathbb{O}, \tilde{\mathbb{O}}$, respectively. Since they preserve the identity and the conjugation they must be subgroups of $\mathrm{O}(7), \mathrm{O}(3, 4)$, respectively.

From now on we will use V to denote either \mathbb{R}^7 or $\mathbb{R}^{3,4}$, and \langle, \rangle_V will be the (pseudo-)scalar product associated to V . Moreover, we identify V with the imaginary part of $\mathbb{O}(V)$, where

$$\mathbb{O}(V) := \begin{cases} \mathbb{O} & \text{if } V = \mathbb{R}^7 \\ \tilde{\mathbb{O}} & \text{if } V = \mathbb{R}^{3,4} \end{cases}$$

$$\mathbb{O}(V) := \begin{cases} \mathbb{O} & \text{if } V = \mathbb{R}^7 \\ \tilde{\mathbb{O}} & \text{if } V = \mathbb{R}^{3,4} \end{cases}$$

and $G_2(V)$ will be the automorphisms group of $\mathbb{O}(V)$. In particular, $G_2(V)$ preserves the so-called *associative 3-form* on V

$$\phi_V(x, y, z) := \langle x, yz \rangle, \quad \phi_V \in \bigwedge^3 V^*$$

and the *coassociative 4-form*

$$\psi_V(x, y, z, w) := \langle x, y(zw) - w(z y) \rangle, \quad \psi_V \in \bigwedge^4 V^*$$

$G_2(V)$ preserves $\phi \wedge \psi = d\text{vol}(V)$ and, hence, it is a subgroup of $\text{SO}(V)$.

Moreover, it can be shown that $G_2(V)$ is the stabilizer of ϕ_V in $\text{GL}(7)$. The equation

$$g^* \phi = \phi$$

implies that $G_2(V)$ is a 14-dimensional Lie group.

Taking the derivative of the last relation, we obtain the relations which define the Lie algebra $\mathfrak{g}_2(V) \subset \mathfrak{so}(V)$:

$$\begin{cases} a_{12} + a_{47} - a_{56} &= 0 \\ a_{13} - a_{46} - a_{57} &= 0 \\ a_{14} - a_{27} - \epsilon_V a_{36} &= 0 \\ a_{15} + a_{26} - \epsilon_V a_{37} &= 0 \\ a_{16} - a_{25} + \epsilon_V a_{34} &= 0 \\ a_{17} + a_{24} + \epsilon_V a_{35} &= 0 \\ a_{23} + a_{45} - a_{67} &= 0 \end{cases} \quad (\{a_{ij}\} \in \mathfrak{so}(V))$$

where

$$\epsilon_V := \begin{cases} +1 & \text{if } V = \mathbb{R}^7, \\ -1 & \text{if } V = \mathbb{R}^{3,4}. \end{cases}$$

Furthermore, using the Moufang identities it is easy to prove

Theorem 7.4. *A matrix $A = (a_1|a_2|a_3|a_4|a_5|a_6|a_7) \in \text{GL}(7)$ belongs to $G_2(V)$ if and only if*

$$\begin{cases} a_4 a_5 &= \epsilon_V a_1 \\ a_4 a_6 &= \epsilon_V a_2 \\ a_4 a_7 &= \epsilon_V a_3 \\ a_4 a_5 + a_6 a_7 &= 0 \end{cases}$$

and (a_4, a_5, a_6, a_7) is an orthogonal 4-frame with $\|a_4\| = \|a_5\| = \|a_6\| = \|a_7\| = \epsilon_V$.

7.2. The canonical 1-dimensional reduction. We are going to link the (split) octonions to a particular action on $\text{SO}(V)$.

First of all, note that the scalar product on V defines a linear map

$$F : \mathfrak{so}(V) \rightarrow \bigwedge^2 V^*$$

$$A \mapsto \langle A(\cdot), \cdot \rangle$$

which is an isomorphism. In particular, every two-form ω on V defines a left action on $\text{SO}(V)$ (and a QK action on $\text{SO}(V)/\text{SO}(3) \times \text{SO}(4)$). In this case, the moment map can be expressed in terms of the 2-form ω :

$$\mu_{F^{-1}(\omega)}(g) = \begin{pmatrix} \omega(f_1, f_2) + \omega(f_3, f_4) \\ \omega(f_1, f_3) - \omega(f_2, f_4) \\ \omega(f_1, f_4) + \omega(f_2, f_3) \end{pmatrix},$$

where f_1, f_2, f_3, f_4 are the last four columns of $g \in \text{SO}(V)$. In particular, the quaternionic zero locus can be described as the set of all the euclidean 4-planes in V such that the restriction of ω on them satisfies $*\omega = -\omega$.

Now, fix $x \in V$ and set $\omega = i_x \phi_V$, where ϕ_V is the associative 3-form on V .

In this case, the moment map equations are

$$\langle f_1 f_2 + f_3 f_4, x \rangle = 0,$$

$$\langle f_1 f_3 - f_2 f_4, x \rangle = 0,$$

$$\langle f_1 f_4 + f_2 f_3, x \rangle = 0.$$

Hence we get

Theorem 7.5. *The zero locus Z_G^x of the action generated by $A_x := F^{-1}(i_x \phi)$ contains the set $T_x \cdot G_2(V) \cdot \text{SO}(3)$, where T_x is the group generated by A_x .*

Proof: Z_G^x contains the set Σ defined by equations

$$f_1 f_2 + f_3 f_4 = 0,$$

$$f_1 f_3 - f_2 f_4 = 0,$$

$$f_1 f_4 + f_2 f_3 = 0.$$

The Moufang identities imply that each of these equations is sufficient to describe Σ , and the statement follows from the previous theorem. \square

We now search for the elements $x \in V$ such that the equality holds, i.e.

$$Z_G^x = T_x \cdot G_2(V) \cdot \text{SO}(3)$$

From the relation $A_{\lambda x} = \lambda A_x$ it follows that we can restrict x to the sphere $S(V) \subset V$.

Moreover, for any $g \in G_2(V)$ we get

$$A_{g(x)} = \text{Ad}_g^{-1}(A_x).$$

Hence,

- if $V = \mathbb{R}^7$ we have just one case, since G_2 acts transitively on S^6 ;
- if $V = \mathbb{R}^{3,4}$, $S(V) = S^+ \cup S^- \cup K - \{0\}$, and each component is an orbit of $G_{2(2)}$, so we have *three* cases, i.e. $x \in S^+$ (the **time-like case**), $x \in S^-$ (the **space-like case**) or $x \in K - \{0\}$ (the **light-like case**).

Furthermore, Z_G^x is acted on by the group

$$U_x := T_x \ltimes H_x,$$

where H_x is the isotropy subgroup of x in $G_2(V)$.

Theorem 7.6. *Suppose either $x \in S^6$ (in the compact case) or $x \in S^+$ (in the noncompact case). Then,*

$$Z_G^x = T_x \cdot G_2(V) \cdot \text{SO}(3)$$

and

$$\mathcal{R}_G^x = \mathbb{Z}_3 \setminus G_2(V) \cdot \text{SO}(3).$$

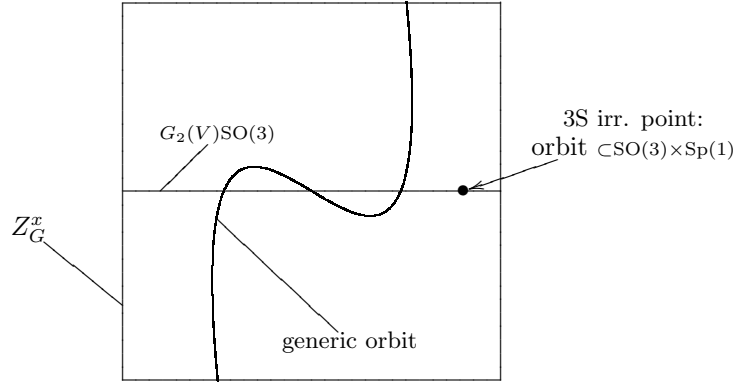
By taking the quotients of \mathcal{R}_G^x we get the following 3-Sasakian, twistor and QK reductions:

$$\mathbb{Z}_3 \setminus G_2(V)/\mathrm{Sp}(1),$$

$$\mathbb{Z}_3 \setminus G_2(V)/\mathrm{Sp}(1)\mathrm{U}(1),$$

$$\mathbb{Z}_3 \setminus G_2(V)/\mathrm{SO}(4).$$

In particular, the QK reduction in the compact case as been already found in [KS93]. So, the reduced diamond diagram is covered (with branches) by the associated diamond diagram to the Wolf space $G_2(V)/\mathrm{SO}(4)$. Note that the action of T_x at the 3-Sasakian level is not locally free, but only *quasi-free*. In fact, this is a typical case of a not-locally free action which produces anyway an orbifold structure on the reduction, thanks to existence of a *section*:



The branched locus of the reduction consists, at the level of the group, of the set of all the (3-Sasakian) irregular points in $\mathrm{SO}(V)$.

In order to describe it, we can consider its quotient by $\mathrm{SO}(3)$, which is a subset of the Stiefel manifold $\mathcal{V}_{0,4}(V)$.

Theorem 7.7. *Let us define $z_r := e_{2r} + ie_{2r+1} \in V \otimes \mathbb{C}$. Then, the quotient of the fixed points set by $\mathrm{SO}(3)$ is*

$$(\mathrm{U}(z_1, z_2, z_3) \cdot \mathrm{Sp}(1)_-)/\mathrm{U}(z_1) \subset \mathcal{V}_{0,4}(V)$$

Hence, the 3-Sasakian branch locus is

$$\mathrm{U}(z_1, z_2, z_3)/(\mathrm{U}(z_1) \times \Delta(\mathrm{U}(z_2, z_3))),$$

where $\Delta(\mathrm{U}(z_2, z_3))$ denotes the subgroup of $\mathrm{U}(z_2, z_3)$ which consists of the complex multiples of the identity. Finally, the quaternionic branch locus is

$$\mathrm{U}(z_1, z_2, z_3)/(\mathrm{U}(z_1) \times \mathrm{U}(z_2, z_3)).$$

Now, let us consider the noncompact case with $x \in S^+ \cup K - \{0\}$. In this case, the picture is completely different! In fact, the set $G_{2(2)}\mathrm{SO}(3)$ isn't a section anymore, but the action is proper and free, so we obtain (nonempty) *manifolds*. As we'll see, we get whole families of *manifolds* carrying out the reduction.

7.3. The adjoint orbits of $\mathrm{SO}(3,4)$ and the associated reductions. In the following we specialize a general construction in [BC77] to the case of orthogonal groups. An analogous application of the construction (in the case of symplectic groups) can be found in [BCGP05].

For any complex vector space V and a symmetric, complex-valued, non-degenerate bilinear form τ let $\mathrm{O}(V, \tau)$ be the group of all the (linear, complex) automorphisms of V which preserve the bilinear form τ and let $\mathfrak{o}(V, \tau)$ be its Lie algebra, which is the algebra of all skew-selfadjoint complex endomorphisms of V .

Let $\sigma : V \rightarrow V$ be an anti-endomorphism (i.e., $\sigma(\alpha v) = \overline{\alpha}\sigma(v)$ for any $\alpha \in \mathbb{C}$, $v \in V$) such that $\sigma^2 = 1$, $\tau^\sigma = \overline{\tau}$.

If $V_\sigma := \{v \in V : \sigma(v) = v\}$ and τ_σ is the restriction of τ on this real subspace, then τ_σ is real and non-degenerate. The group

$$\mathrm{O}(V, \tau, \sigma) := \{g \in \mathrm{O}(V, \tau) : g^\sigma = g\}$$

can be identified (through restriction to V_σ) with the group of real automorphisms of V_σ which preserve τ_σ , and so it is isomorphic to $\mathrm{O}(k, l)$, where (k, l) is the signature of τ_σ . Let $\mathfrak{o}(V, \tau, \sigma)$ be its Lie algebra.

Let $A \in \mathfrak{o}(V, \tau, [\sigma])$, $A' \in \mathfrak{o}(V', \tau', [\sigma'])$. We say A, A' to be *equivalent* if there exists an isomorphism $\phi : V \rightarrow V'$ such that $\tau' = \tau^\phi$, $[\sigma' = \phi\sigma\phi^{-1}]$ and $A' = \phi A \phi^{-1}$.

This definition is actually an equivalence relation and its classes are called *types*:

$$\Delta := [A].$$

Furthermore, ϕ as above defines an isomorphism between $\mathrm{O}(V, \tau, [\sigma])$ and $\mathrm{O}(V', \tau', [\sigma'])$, and in the case $(V, \tau, [\sigma]) = (V', \tau', [\sigma'])$, A, A' are equivalent if and only if they belong to the same adjoint orbit. It's straightforward to define the *sum* of two types:

$$\Delta, \Delta' \mapsto \Delta \oplus \Delta'.$$

Theorem 7.8. *Every type can be written (up to the order) in a unique way as a sum of indecomposable types.*

Every $A \in \mathfrak{o}(V, \tau, [\sigma])$ can be decomposed in a unique way as a sum

$$A = S + N,$$

where $S, N \in \mathfrak{o}(V, \tau, [\sigma])$, $[S, N] = 0$ and are respectively semisimple and nilpotent. The nonnegative integer k such that $N^k \neq 0$, $N^{k+1} = 0$ is called *the height* of A . It is an invariant of the type $\Delta = [A]$ and we denote it with $\mathrm{ht}(\Delta)$. (Another invariant is the *dimension* of the type.) A type with null height is called *semisimple*.

There are two kinds of semisimple indecomposable types for $\mathrm{O}(V, \tau)$:

- $\Delta(\zeta, -\zeta)$, $\zeta \neq 0$: it is \mathbb{C} -generated by two nonorthogonal vectors v, w of norm 0, eigenvectors of S with respect to the eigenvectors $\zeta, -\zeta$;
- $\Delta(0)$, if $S = 0$: it is \mathbb{C} -generated by a vector of norm 1.

In the case of $\mathrm{O}(V, \tau, \sigma)$ we get more semisimple indecomposable types, namely

- $\Delta(\zeta, -\zeta, \overline{\zeta}, -\overline{\zeta})$ with $\overline{\zeta} \neq \pm\zeta$. This *real* type can be construct as follows: take the complex type

$$\Delta(\zeta, -\zeta) \oplus \Delta(\overline{\zeta}, -\overline{\zeta})$$

which is \mathbb{C} -generated by v_1, v_2, v_3, v_4 such that

$$\tau(v_h, v_l) = \begin{cases} 1 & \text{if } \{h, l\} = \{1, 2\} \text{ or } \{3, 4\}; \\ 0 & \text{otherwise} \end{cases}$$

and which are eigenvectors of $\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta}$, respectively. Furthermore, after rescaling we can suppose V_σ is \mathbb{R} -generated by the orthogonal vectors

$$\begin{aligned} w_1 &= \frac{1}{2}(v_1 + v_2 + v_3 + v_4), & \|w_1\| &= 1, \\ w_2 &= \frac{i}{2}(v_1 - v_2 - v_3 + v_4), & \|w_2\| &= 1, \\ w_3 &= \frac{1}{2}(v_1 - v_2 + v_3 - v_4), & \|w_3\| &= -1, \\ w_4 &= \frac{i}{2}(v_1 + v_2 - v_3 - v_4), & \|w_4\| &= -1. \end{aligned}$$

Hence, we get

$$\begin{cases} Sw_1 &= aw_3 + bw_2 \\ Sw_2 &= aw_4 - bw_1 \\ Sw_3 &= aw_1 + bw_4 \\ Sw_4 &= aw_2 - bw_3, \end{cases}$$

where $\zeta = a + ib$, $a, b \in \mathbb{R}$.

- $\Delta(\zeta, -\zeta)$ with $\bar{\zeta} = \zeta \neq 0$: Let v_1, v_2 be the generators of the respective complex type. In this case V_σ is \mathbb{R} -generated by v_1, v_2 and also by the orthogonal vectors

$$\begin{aligned} w_1 &:= \frac{1}{\sqrt{2}}(v_1 + v_2), & \|w_1\| &= 1, \\ w_2 &:= \frac{1}{\sqrt{2}}(v_1 - v_2), & \|w_2\| &= -1. \end{aligned}$$

Hence, we get

$$\begin{cases} Sw_1 &= aw_2 \\ Sw_2 &= aw_1, \end{cases}$$

where $\zeta = a + i0$, $a \in \mathbb{R}$.

- $\Delta^\pm(\zeta, -\zeta)$ with $\bar{\zeta} = -\zeta \neq 0$: As in the previous case, V_σ is generated by two orthogonal vectors w_1, w_2 such that

$$\|w_1\| = \|w_2\| = \pm 1$$

and

$$\begin{cases} Sw_1 &= bw_2 \\ Sw_2 &= -bw_1, \end{cases}$$

where $\zeta = ib$, $b \in \mathbb{R}$.

- $\Delta^\pm(0)$ with $S = 0$: In this case, V_σ is generated by a vector of norm ± 1 .

Let Δ be a generic indecomposable type, of height k . Then $\text{Im } N \leq \text{Ker } N^k$, while we say Δ to be *uniform* if the equality holds. If Δ is uniform, we define

$$\tilde{V} := V/\text{Im } N,$$

$$\tilde{A}(v + \text{Im } N) := Av + \text{Im } N = Sv + \text{Im } N,$$

$$\tilde{\sigma}(v + \text{Im } N) = \sigma(v) + \text{Im } N,$$

$$\tilde{\tau}(v + \text{Im } N, w + \text{Im } N) := \tau(v, N^k(w)).$$

In particular, the bilinear form $\tilde{\tau}$ is non-degenerate and it is symmetric if the height is even: in this case, we get so a semisimple type $\tilde{\Delta}$. In the odd case we no longer get an orthogonal type but a *symplectic* one (i.e., substitute in the previous definitions $O, \mathfrak{o}, \text{symmetric}$ with $Sp, \mathfrak{sp}, \text{skewsymmetric}$). However,

- (1) if Δ is indecomposable, then it is uniform and $\tilde{\Delta}$ is indecomposable;
- (2) if Δ is uniform, then it is uniquely determined by its height and $\tilde{\Delta}$.

In addition we need to describe a classification of the indecomposable semisimple types for $Sp(V, \tau, [\sigma])$.

All the indecomposable semisimple types of $Sp(V, \tau)$ can be denoted again with $\Delta(\zeta, -\zeta)$, where $\zeta \in \mathbb{C}$: in this type, V is \mathbb{C} -generated by two vectors v, w which are eigenvectors of $\zeta, -\zeta$ respectively. Moreover, the indecomposable semisimple types of $Sp(V, \tau, \sigma)$ are

- $\Delta(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$ with $\bar{\zeta} \neq \pm\zeta$: As before, V_σ is \mathbb{R} -generated by w_1, w_2, w_3, w_4 such that

$$\begin{cases} Sw_1 &= aw_3 + bw_2 \\ Sw_2 &= aw_4 - bw_1 \\ Sw_3 &= aw_1 + bw_4 \\ Sw_4 &= aw_2 - bw_3, \end{cases}$$

where $\zeta = a + ib$, $a, b \in \mathbb{R}$. Furthermore, $\tau(w_h, w_l) = -1$ if $(h, l) = (1, 2)$ or $(3, 4)$, and it is equal to 0 otherwise.

- $\Delta(\zeta, -\zeta)$ with $\bar{\zeta} = \zeta \neq 0$: Here V_σ is \mathbb{R} -generated by w_1, w_2 such that

$$\begin{cases} Sw_1 &= aw_2 \\ Sw_2 &= aw_1, \end{cases}$$

where $\zeta = a + i0$, $a \in \mathbb{R}$, and $\tau(w_1, w_2) = -1$.

- $\Delta^\pm(\zeta, -\zeta)$ with $\bar{\zeta} = -\zeta \neq 0$: In this case, V_σ is \mathbb{R} -generated by w_1, w_2 such that

$$\begin{cases} Sw_1 &= bw_2 \\ Sw_2 &= -bw_1, \end{cases}$$

where $\zeta = ib$, and $\tau(w_1, w_2) = \pm 1$.

- $\Delta(0, 0)$ with $S = 0$: In this case, V_σ has real dimension 2.

It remains to describe how Δ can be recovered from $\tilde{\Delta}$ and its height. Let Δ be an indecomposable (\Rightarrow uniform) type of height k . Then

$$V = W \oplus NW \oplus N^2W \oplus \cdots \oplus N^k W$$

such that the subspace W is S -invariant, σ -invariant and $W \perp N^h W$ for $h = 0, \dots, k-1$. Furthermore, the subspaces $N^h W$ are isomorphic and the restriction of S to W (endowed with $\tilde{\tau}, \tilde{\sigma}$) represents the type $\tilde{\Delta}$.

Conversely, let $\Delta = [S]$ a semisimple indecomposable type and k a positive integer. If $S \in \mathfrak{g}(V, \tau, \sigma)$, let us define

$$V_k := \bigoplus_{i=0}^{k-1} N^i W, \quad S_k := \bigoplus_{i=0}^{k-1} N^i S, \quad \sigma_k := \bigoplus_{i=0}^{k-1} N^i \sigma.$$

Furthermore, let N_k be the endomorphism of V_k which does shift the components, namely

$$N_k(v_0, v_1, \dots, v_k) := (0, v_0, \dots, v_{k-1})$$

and let be τ_k the unique bilinear form on V_k with respect to which N is skewsymmetric and such that, for every $u, v \in W$,

$$\tau_k(u, N^i v) = \begin{cases} \tau(u, v) & \text{if } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\Delta_k := [S_k + N_k]$ is an indecomposable type of height k such that $\widetilde{\Delta}_k = \Delta$. Below is the list of all the indecomposable, orthogonal types of dimension ≤ 7 :

| Δ | | $\dim(\Delta)$ | signature |
|--|--|----------------|-----------|
| $\Delta_6^-(0)$ | | 7 | (4, 3) |
| $\Delta_6^+(0)$ | | 7 | (3, 4) |
| $\Delta_2^-(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 6 | (4, 2) |
| $\Delta_2(\zeta, -\zeta)$ | $0 \neq \zeta \in \mathbb{R}$ | 6 | (3, 3) |
| $\Delta_2^+(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 6 | (2, 4) |
| $\Delta_4^+(0)$ | | 5 | (3, 2) |
| $\Delta_4^-(0)$ | | 5 | (2, 3) |
| $\Delta_0(\zeta, -\zeta, \zeta, -\zeta)$ | $\zeta \in \mathbb{C} - (\mathbb{R} \cup i\mathbb{R})$ | 4 | (2, 2) |
| $\Delta_1^-(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 4 | (2, 2) |
| $\Delta_1(\zeta, -\zeta)$ | $\zeta \in \mathbb{R}$ | 4 | (2, 2) |
| $\Delta_1^+(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 4 | (2, 2) |
| $\Delta_2^-(0)$ | | 3 | (2, 1) |
| $\Delta_2^+(0)$ | | 3 | (1, 2) |
| $\Delta_0^+(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 2 | (2, 0) |
| $\Delta_0(\zeta, -\zeta)$ | $0 \neq \zeta \in \mathbb{R}$ | 2 | (1, 1) |
| $\Delta_0^-(\zeta, -\zeta)$ | $0 \neq \zeta \in i\mathbb{R}$ | 2 | (0, 2) |
| $\Delta_0^+(0)$ | | 1 | (1, 0) |
| $\Delta_0^-(0)$ | | 1 | (0, 1) |

By combining them together, we get a list of 48 orthogonal types with signature (3, 4).

Furthermore, by setting

$$\begin{aligned} \Delta_2^+(0, 0) &:= 2\Delta_2^+(0) \\ \Delta_2(0, 0) &:= \Delta_2^+(0) \oplus \Delta_2^-(0) \\ \Delta_0(a, -a, a, -a) &:= 2\Delta_0(a, -a) \\ \Delta_0(ib, -ib, -ib, ib) &:= \Delta_0^+(ib, -ib) \oplus \Delta_0^-(ib, -ib) \\ \Delta_0(0, 0) &:= \Delta_0^+(0) \oplus \Delta_0^-(0) \\ \Delta_0^\pm(0, 0) &:= 2\Delta_0^\pm(0) \end{aligned}$$

for $a, b \in \mathbb{R}$, we obtain only 24 families:

In particular,

$$A_x \in \begin{cases} \Delta_0^+(i, -i) \oplus 2\Delta_0^-(i, i) \oplus \Delta_0^+(0) & \text{if } \|x\| = 1, \\ 3\Delta_0(1, -1) \oplus \Delta_0^-(0) & \text{if } \|x\| = -1, \\ \Delta_1(0, 0) \oplus \Delta_2^+(0) & \text{if } \|x\| = 0. \end{cases}$$

| Δ | $\text{ht}(\Delta)$ | # params. | Name |
|--|---------------------|-----------|-------------------|
| $\Delta_6^+(0)$ | 6 | 0 | I ₁ |
| $\Delta_4^+(0) \oplus \Delta_0^-(\zeta, -\zeta)$ | 4 | 1 | II ₁ |
| $\Delta_4^-(0) \oplus \Delta_0(\zeta, -\zeta)$ | 4 | 1 | II ₂ |
| $\Delta_2(\zeta, -\zeta) \oplus \Delta_0^-(0)$ | 2 | 1 | II ₃ |
| $\Delta_2^+(\zeta, -\zeta) \oplus \Delta_0^+(0)$ | 2 | 1 | II ₄ |
| $\Delta_1^-(\zeta, -\zeta) \oplus \Delta_2^+(0)$ | 2 | 1 | II ₅ |
| $\Delta_1(\zeta, -\zeta) \oplus \Delta_2^+(0)$ | 2 | 1 | II ₆ |
| $\Delta_1^+(\zeta, -\zeta) \oplus \Delta_2^+(0)$ | 2 | 1 | II ₇ |
| $\Delta_0(\zeta, -\zeta, \zeta, -\zeta) \oplus \Delta_2^+(0)$ | 2 | 2 | III ₁ |
| $\Delta_2^-(0) \oplus \Delta_0(\zeta_1, -\zeta_1) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 2 | 2 | III ₂ |
| $\Delta_0^+(\zeta_1, -\zeta_1) \oplus \Delta_2^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 2 | 2 | III ₃ |
| $\Delta_2^+(0) \oplus \Delta_0(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2)$ | 2 | 2 | III ₄ |
| $\Delta_1^-(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^-(0)$ | 1 | 2 | III ₅ |
| $\Delta_1(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^-(0)$ | 1 | 2 | III ₆ |
| $\Delta_1^+(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^-(0)$ | 1 | 2 | III ₇ |
| $\Delta_1^-(\zeta_1, -\zeta_1) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 1 | 2 | III ₈ |
| $\Delta_1(\zeta_1, -\zeta_1) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 1 | 2 | III ₉ |
| $\Delta_1^+(\zeta_1, -\zeta_1) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 1 | 2 | III ₁₀ |
| $\Delta_0(\zeta_1, -\zeta_1, \zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^-(0)$ | 0 | 3 | IV ₁ |
| $\Delta_0(\zeta_1, -\zeta_1, \zeta_1, -\zeta_1) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2)$ | 0 | 3 | IV ₂ |
| $\Delta_0^+(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^-(\zeta_3, -\zeta_3) \oplus \Delta_0^-(0)$ | 0 | 3 | IV ₃ |
| $\Delta_0^+(\zeta_1, -\zeta_1) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_2, -\zeta_2) \oplus \Delta_0^-(\zeta_3, -\zeta_3)$ | 0 | 3 | IV ₄ |
| $\Delta_0(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0(\zeta_3, -\zeta_3) \oplus \Delta_0^-(0)$ | 0 | 3 | IV ₅ |
| $\Delta_0(\zeta_1, -\zeta_1) \oplus \Delta_0(\zeta_2, -\zeta_2) \oplus \Delta_0^+(0) \oplus \Delta_0^-(\zeta_3, -\zeta_3)$ | 0 | 3 | IV ₆ |

One can easily show

Theorem 7.9. *Let $A \in \mathfrak{so}(3, 4)$, belonging to the adjoint orbit $[A]$. If $[A] \notin \text{IV}_4$, then the quaternionic action generated by A is proper and free at all levels.*

Proof: If $[A] \notin \text{IV}_4$, the matrix $\exp(tA) \in \text{SO}(3, 4)$ contains polynomial expressions and/or hyperbolic functions, so

$$\lim_{t \rightarrow \pm\infty} \|\exp(tA)\| = +\infty,$$

where $\|\cdot\|$ denotes the euclidean norm of \mathbb{R}^{49} . So, the group generated by A is the Lie subgroup of $\text{SO}(3, 4)$ and the action is proper on the group level. Furthermore, the restriction of the action on Z_G is proper, as well as the actions induced on the 3-Sasakian, the twistor, and the QK level.

Since the group generated by A is isomorphic to \mathbb{R} , the freeness follows from the properness. \square

Corollary 7.10. *Let $A \in \mathfrak{so}(3, 4)$, belonging to the adjoint orbit $[A]$. If $[A] \notin \text{IV}_4$, then the reduction associated to A produces manifolds.*

Furthermore, from a straightforward computation of the centralizers, it follows that the reduced diamond diagrams admit (at least) two commuting Killing vector fields.

Theorem 7.11. *Let $A \in \mathfrak{so}(3, 4)$, belonging to the adjoint orbit $[A] = \text{IV}_4(a, b, c)$. The quaternionic action generated by A is proper if and only if the triple (a, b, c)*

is commensurable. In this case, if $0 \notin \{a, b, c\}$ we can suppose a, b, c to be positive integers with $\gcd(a, b, c) = 1$. Furthermore, there are no 3-Sasakian irregular points if and only if a, b, c are distinct.

Analogously to the compact case, it is possible to give conditions to the coefficients (a, b, c) in order to obtain a free 3-sasakian action (see [BGP02]). It seems hard to can find stronger conditions for the QK action.

Remark 7.12. *The previous results can be used to study the 1-dimensional QK reductions in the case of the Wolf space $G_{2(2)}/\mathrm{SO}(4)$. In fact, any element $v \in \mathfrak{g}_{2(2)}$ determines an adjoint orbit in $\mathfrak{so}(3, 4) \triangleleft \mathfrak{so}(7, 1)$. So, $\Delta \notin \mathrm{IV}_4$ if and only if the adjoint orbit of v in $\mathfrak{g}_{2(2)}$ doesn't intersect $\mathfrak{so}(4) \subset \mathfrak{g}_{2(2)}$ and in this case the reduction produces self-dual, Einstein 4-manifolds with negative sectional curvature and few symmetries.*

Open problems:

- In the non- IV_4 cases, classify the obtained manifolds. In particular, is it true that the quaternionic Killing algebra is the quotient $\mathfrak{cent}(\mathfrak{t})/\mathfrak{t}$? Does it coincide with the Killing algebra of the whole reduced diamond diagram? Are there algebraic invariants which classify this manifolds?
- In the IV_4 case, classify the obtained orbifolds. In the case of two equal parameters, does the action admit sections (like in the case $(1, 1, 1)$)?
- Do the same for 2-dimensional toric actions. How to classify the conjugacy classes of abelian subalgebras (of dimension > 1) of classic Lie algebras?
- Repeat the whole construction for the other Wolf spaces. In particular, we want to find 4-dimensional manifolds with few symmetries, so would be better to start from low-dimensional, noncompact Wolf spaces, for example $G_{2(2)}/\mathrm{SO}(4)$. How to classify the adjoint orbits of $G_{2(2)}$ and in general, of an exceptional Lie group?
- In the case of non-Riemannian Wolf spaces, reduction process can encounter a further problem: in fact, the *reduced metric* may have some kernel.

REFERENCES

- [AC97] D. V. Alekseevskii and V. Cortés, *Homogeneous quaternionic khler manifolds of uni-modular group*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, 217–229. MR **1456262** (98i:53062)
- [AC05] ———, *Classification of pseudo-Riemannian symmetric spaces of quaternionic Kähler type*, Lie groups and invariant theory, Amer. Math. Soc. Transl. Ser. 2, vol. 213, Amer. Math. Soc., Providence, RI, 2005, pp. 33–62. MR 2140713 (Review)
- [Ale68] D. V. Alekseevskii, *Riemannian spaces with unusual holonomy groups*, Funkcional. Anal. i Priložen **2** (1968), no. 2, 1–10. MR 37 #6868
- [Ale75] ———, *Classification of quaternionic spaces with transitive solvable group of motions*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 2, 315–362, 472. MR 53 #6465
- [AM78] R. Abraham and J. E. Marsden, *Foundations of mechanics*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978, Second edition, revised and enlarged, With the assistance of Tudor Rațiu and Richard Cushman. MR **81e:58025**
- [Bat04] L. Bates, *A symmetry completeness criterion for second-order differential equations*, Proc. AMS **132** (2004), no. 6, 1785–1786. MR **2051142** (2005b:34076)
- [BC77] N. Burgoyne and R. Cushman, *Conjugacy classes in linear groups*, J. Algebra **44** (1977), no. 2, 339–362. MR 55 #5761
- [BCGP05] C. P. Boyer, D. M. J. Calderbank, K. Galicki, and P. Piccinni, *Toric self-dual Einstein metrics as quotients*, Comm. Math. Phys. **253** (2005), no. 2, 337–370. MR 2140252

- [Ber55] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330. MR 18,149a
- [BGM93] C. P. Boyer, K. Galicki, and B. M. Mann, *Quaternionic reduction and Einstein manifolds*, Comm. Anal. Geom. **1** (1993), no. 2, 229–279. MR **95c**:53056
- [BGM94] ———, *The geometry and topology of 3-Sasakian manifolds*, J. Reine Angew. Math. **455** (1994), 183–220. MR **96e**:53057
- [BGP02] C. P. Boyer, K. Galicki, and P. Piccinni, *3-Sasakian geometry, nilpotent orbits, and exceptional quotients*, Ann. Global Anal. Geom. **21** (2002), no. 1, 85–110. MR **2003d**:53076
- [Gal86] K. Galicki, *Quaternionic Kähler and hyper-Kähler nonlinear σ -models*, Nuclear Phys. B **271** (1986), no. 2, 402–416. MR **88m**:53087
- [Gal87] ———, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Comm. Math. Phys. **108** (1987), no. 1, 117–138. MR **88f**:53088
- [GL88] K. Galicki and H. B. Lawson, Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), no. 1, 1–21. MR **89m**:53075
- [Har90] F. R. Harvey, *Spinors and calibrations*, Perspectives in Mathematics, vol. 9, Academic Press, Boston, 1990.
- [Kon75] M. Konishi, *On manifolds with Sasakian 3-structure over quaternion Kaehler manifolds*, Kodai Math. Sem. Rep. **26** (1974/75), 194–200. MR 51 #13951
- [KS93] P. Z. Kobak and A. Swann, *Quaternionic geometry of a nilpotent variety*, Math. Ann. **297** (1993), no. 4, 747–764. MR **94j**:53057
- [MW74] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Mathematical Phys. **5** (1974), no. 1, 121–130. MR 0402819 (53 #6633)
- [Sal82] S. M. Salamon, *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), no. 1, 143–171. MR **83k**:53054
- [Swa91] A. Swann, *Hyper-Kähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), no. 3, 421–450. MR **92c**:53030
- [Wol65] J. A. Wolf, *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033–1047. MR 32 #3020

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